# Universally Reflexive Algebras*

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ABSTRACT

Finite dimensional algebras whose representations are all reflexive are characterized as those over which the homomorphisms between any pair of indecomposable projective modules form a vector space of dimension at most one. This includes incidence algebras over finite preordered sets.

If $R$ is an algebra over a field $K$, a left $R$-module $M$ is called *reflexive* if the only $K$-linear maps $M \to M$ which leave invariant every $R$-submodule of $M$ are given by multiplication by elements of $R$. This notion is derived from one due to Halmos [6] and has been studied in the context of operator algebras, for example, by Azoff [2]. In [4] Habibi and Gustafson showed that

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if $R$ is a basic indecomposable split serial ring, then every faithful left $R$-module is reflexive provided that $R$ has more than one isomorphism class of indecomposable injective projective modules. In this note we characterize the finite dimensional algebras over which every left module is reflexive. As a bonus, the characterization turns out to be right-left symmetric, so these are just the algebras with every right module reflexive. Our characterization applies, in particular, to incidence algebras and (as do the results in [4]) to indecomposable split serial algebras with Loewy length less than the number of indecomposable projective modules.

The proof refers to results in [3], where the notion of a reflexive bimodule is introduced. If ${}_R M_\Delta$ is any bimodule, we define

$$\operatorname{alglat}({}_R M_\Delta) = \{\alpha \in \operatorname{end} M_\Delta \mid \alpha N \subseteq N \text{ for all } {}_R N \subseteq {}_R M\}.$$

This is a ring and contains the image $\lambda_M(R)$ of $R$ under the ring homomorphism $\lambda_M : R \to \operatorname{end} M_\Delta$, where $\lambda_M(r)$ is multiplication by $r$. Then ${}_R M_\Delta$ is called a *reflexive bimodule* if $\lambda_M(R) = \operatorname{alglat}({}_R M_\Delta)$. If $R$ is a $K$-algebra, then a left module ${}_R M$ is a bimodule ${}_R M_K$ (where $km = mk$), and this notion of reflexivity reduces to the algebra version. The efficacy of the more general idea was demonstrated in [3] where, for example, we showed that reflexivity is preserved under Morita equivalence, and we extended results of Hadwin and Kerr [5] by proving that a commutative artinian ring is quasi-Frobenius if and only if every faithful module ${}_R M$ is reflexive as an $R$-$R$-bimodule. Here we employ the results of [3] to prove the following theorem. Terms not defined here can be found in [1].

THEOREM. *Let $R$ be a finite dimensional algebra over a field $K$, and let $e_1, \ldots, e_n$ be a basic set of primitive idempotents for $R$. Then the following statements are equivalent*:

(a) *Every left $R$-module is reflexive.*

(b) *If $P$ is an indecomposable projective left $R$-module, then every epimorphic image of $P \oplus P$ is reflexive.*

(c) *If $P$ and $Q$ are indecomposable projective left $R$-modules, then* $|\operatorname{Hom}_R(P, Q) : K| \leqslant 1$.

(d) $|e_i R e_j : K| \leqslant 1$ *for all* $i, j \in \{1, \ldots, n\}$.

(e) *Every right $R$-module is reflexive.*

*Proof.* Recalling that reflexivity of $K$-algebra modules is preserved under Morita equivalence [3], we assume $R$ is a basic algebra with radical $J$.

(a) $\Rightarrow$ (b): This is obvious.

(c) $\Leftrightarrow$ (d): These conditions are equivalent because $Re_1, \ldots, Re_n$ represent all the indecomposable projective left $R$-modules, and $\mathrm{Hom}_R(Re_i, Re_j) \cong e_i Re_j$.

(b) $\Rightarrow$ (d): Assuming (b), each $Re_i$ is reflexive, so by [3, Proposition 3.5] $R$ is split as well as basic and $e_i Je_i = 0$. Thus ${}_K Re_i = Ke_i \oplus Je_i$ and $e_i Re_i = Ke_i$ for $i = 1, \ldots, n$. Suppose that $|e_i Re_j : K| \geqslant 2$ for some $i \neq j$. Then the factor module $Y = Re_j / Je_i Re_j$ of $Re_j$ must contain two distinct copies of $Re_i / Je_i$, i.e., $Y$ is a local module

$$Y = Re_j z \qquad \text{with generator} \quad z = e_j z$$

and $K$-independent elements $s, t \in Y$ such that

$$Ks = Re_i s \quad \text{and} \quad Kt = Re_i t.$$

Our nonreflexive module is

$$M = (Y \times Y)/L$$

with

$$L = K(s, 0) \oplus K(t, -s).$$

Here for each $y \in Y$ we write

$$\bar{y} = (y, 0) + L \quad \text{and} \quad \tilde{y} = (0, y) + L,$$

so that

$$\bar{s} = 0 \quad \text{and} \quad \bar{t} = \tilde{s}.$$

Moreover, letting $\bar{Y} = Re_j \bar{z}$ and $\tilde{Y} = Re_j \tilde{z}$, we see that $\tilde{Y} \cong Y$ and $\bar{Y} \cong Y/Ks$, and that there is an $R$-epimorphism

$$\varphi : \tilde{Y} \to \bar{Y}$$

via

$$\varphi : \tilde{y} \to \bar{y}.$$

Thus if $r \in R$ and $k \in K$ with

$$r\tilde{z} = k\tilde{t},$$

then

$$r\bar{z} = r\varphi(\bar{z}) = \varphi(r\bar{z}) = k\bar{t} = k\bar{s}.$$

Since clearly

$${}_K M = K\bar{z} \oplus K\tilde{z} \oplus JM,$$

there is a linear transformation $\alpha \in \operatorname{end}(M_K)$ such that

$$\alpha(\tilde{z}) = \bar{t} = \tilde{s} \quad \text{and} \quad \alpha(K\bar{z} \oplus JM) = 0,$$

and by the preceding sentence $\alpha \notin \lambda_M(R)$. Suppose now that

$$m = k\bar{z} + l\tilde{z} + w$$

with $k, l \in K$ and $w \in JM$, and choose $u, v \in e_i R e_j$ such that

$$uz = s \quad \text{and} \quad vz = t.$$

Then noting that $uJM = ue_j Je_j \bar{z} + ue_j Je_j \tilde{z} = 0 = vJM$, we see that if $l = 0$ then

$$vm = kv\bar{z} = k\bar{t} = \alpha(m),$$

and if $l \neq 0$ then

$$kl^{-1}um = kl^{-1}u\tilde{z} + ku\bar{z} = kl^{-1}\tilde{s} + k\bar{s} = \alpha(m),$$

so that $\alpha \in \operatorname{alglat}(M) \setminus \lambda_M(R)$.

(d) $\Rightarrow$ (a): Assuming (d), let $M$ be any left $R$-module and let $A = \operatorname{alglat}(M)$. Since (d) is inherited by $R/\operatorname{ann}(M)$, we may identify $R$ with $\lambda_M(R) \subseteq A$. Having done so, we first note that if $e_i R e_j = 0$, then also $e_i A e_j = 0$, because if $\alpha \in A$ and $m \in M$, then $e_i \alpha e_j m = re_j m = e_i re_j m$ for some $r \in R$. In each of the remaining $e_i R e_j \neq 0$ choose $c_{ij}$ such that

$$Kc_{ij} = e_i R e_j.$$

Then, if $\alpha = e_i \alpha e_j \in e_i A e_j$ and $m \in M$, there is a $k \in K$ such that

$$\alpha m = k c_{ij} m.$$

Suppose that $0 \neq \alpha \in e_i A e_j$. If $x, x' \in M$ with $\alpha x \neq 0$ and $x' \neq 0$, then there must be elements $k, k' \in K$ such that

$$\alpha x = k c_{ij} x \quad \text{and} \quad \alpha x' = k' c_{ij} x'$$

and an $l \in K$ such that

$$\alpha(x + x') = l c_{ij} x + l c_{ij} x'.$$

If $K c_{ij} x \cap K c_{ij} x' = 0$ then

$$k = l = k'.$$

If, on the other hand, there is an element $p \in K$ such that $p c_{ij} x' = c_{ij} x$, then for some $d \in K$ we have

$$\begin{aligned} \alpha(x - px') &= d c_{ij}(x - px') \\ &= d(c_{ij} x - p c_{ij} x') = 0, \end{aligned}$$

so that

$$\begin{aligned} (k - k') c_{ij} x &= k c_{ij} x - k' p c_{ij} x' \\ &= \alpha x - p \alpha x' \\ &= \alpha(x - px') = 0. \end{aligned}$$

Thus $\alpha$ is left multiplication by $k c_{ij} \in e_i R e_j$, and it follows that $A = R$.

(e) $\Leftrightarrow$ (d): This follows because (d) is right-left symmetric and is equivalent to (a). ■

Recall that if $X = \{x_1, \ldots, x_n\}$ is a finite set preordered by $\leqslant$, the incidence algebra $R = I(X, K)$ is the subalgebra of the $n$ by $n$ matrix ring over the field $K$ spanned by the matrix units $e_{ij}$ such that $x_i \leqslant x_j$. Then since each $e_{ii}Re_{jj}$ is either $Ke_{ij}$ or 0, we have

COROLLARY. *Every representation of an incidence algebra is reflexive.*

REMARKS.

(1) In [3, Theorem 4.1 and Corollary 4.5] we proved that all projective modules are reflexive over split hereditary algebras and over incidence algebras. In addition to improving the latter result rather dramatically, our present theorem shows that the former has little room for improvement. Indeed, the K-algebra of matrices of the form

$$\begin{bmatrix} a & 0 & c \\ 0 & a & d \\ 0 & 0 & b \end{bmatrix}$$

is hereditary but has a (faithful) module $M$ that is not reflexive, because letting $e_1 = e_{11} + e_{22}$ and $e_2 = e_{33}$, one has $|e_1 Re_2 : K| = 2$.

(2) According to the Theorem, all left and all right modules are reflexive over a $K$-algebra $R$ if the finitely generated left $R$-modules are reflexive. One can go from finitely generated left modules to finitely generated right modules directly by observing that if $(\ )^* = \mathrm{Hom}_K(\ ,K)$ denotes the $K$-dual, and $M$ is a finitely generated left $R$-module, then the finitely generated right $R$-module $M^*$ has

$$\mathrm{alglat}(M^*) = \{\alpha^* : M^* \to M^* | \alpha \in \mathrm{alglat}(M)\},$$

and right multiplication by $r \in R$ is

$$\rho_{M^*}(r) = \lambda_M(r)^*.$$

Thus if $M$ is reflexive, then so is $M^*$. In particular, it follows from [3, Theorem 4.1] that every injective module over a split hereditary algebra is reflexive.

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